$(x - a)/d$, Eq. (10) is used to calculate R_m/R_∞, which must be the same as the value picked. This is iterated (once or twice) until the R_m/R_∞ values are the same.

The method given in the foregoing has been compared to the centerline total temperature decay data of Ref. 3 as shown in Fig. 3. For afterburning, ΔH_c was taken as 2200 Btu/lb. These analytical results are seen to be in relatively good agreement with the data.

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A "Membrane" Solution for Axisymmetric Heating of Dome-Shaped Shells of Revolution

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THE following note presents an order of magnitude analysis which leads to a particular solution of the problem of axisymmetric heating of dome-shaped shells of revolution. Such a solution might be termed an approximate analog to Goodier's method in thermoelasticity. The complete solution is constructed by superposing on the particular solution a solution for an edge-loaded shell which is properly adjusted to satisfy the particular boundary conditions of the given problem. This latter solution has been studied extensively, and many approximate solutions are available. The resulting composite solution has the advantage over the more direct methods that the accuracy of the solution in the interior of the shell is quite high.

General Equation

With the notation of Ref. 1 (see Fig. 1), the inclusion of thermal effects modifies the stress-strain law as follows:

$$\sigma_\phi = E\cdot(\epsilon_\phi + \nu\epsilon_\theta)/(1 - \nu^2) - \alpha ET/(1 - \nu)$$
$$\sigma_\theta = E\cdot(\epsilon_\theta + \nu\epsilon_\varphi)/(1 - \nu^2) - \alpha ET/(1 - \nu)$$
$$\tau_{\varphi\theta} = G\gamma_{\varphi\theta}$$

If the temperature variations to be studied are limited to

$$T^{(1)} = S(\varphi) \qquad T^{(2)} = (\zeta/h)\cdot\sigma(\varphi) \quad (-h/2 \leqslant \zeta \leqslant h/2)$$

i.e., either middle surface heating or a temperature gradient, and if the functions $S(\varphi)$, $\sigma(\varphi)$ are slowly varying functions of position, one can develop an order of magnitude analysis for each case on the assumption that the deformation variables will also be slowly varying functions of position. Specifically, derivatives with respect to arc length will be considered to be of the order of magnitude of the quality differentiated divided by a length (R_c) that will be taken equal to a measure of the radius of curvature (R_2), i.e.,

$$\frac{d(N, M)}{ds} = \frac{(N, M)}{R_c}\, 0(1) \qquad \frac{R_2}{R_c} = 0(1)$$

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Upon introducing the following nondimensional variables

$$S(\varphi) = S_c\cdot\tilde{S} \qquad \sigma(\varphi) = \sigma_c\tilde{\sigma}$$
$$\tilde{\epsilon}_i^{(1)} = \frac{\epsilon_{i0}}{\alpha S_c} \qquad \tilde{\epsilon}_i^{(2)} = \frac{12R_c\epsilon_{i0}}{\sigma_c\alpha h}$$
$$\tilde{V}^{(1)} = \frac{V}{\alpha S_c} \qquad \tilde{V}^{(2)} = \frac{12R_cV}{\sigma_c\alpha h}$$
$$\tilde{N}_{ii}^{(1)} = \frac{N_{ii}}{Eh\alpha S_c} \qquad \tilde{N}_{ii}^{(2)} = \frac{12(1 - \nu)R_cN_{ii}}{\sigma_c\alpha Eh^2}$$
$$\tilde{M}_{ii}^{(1)} = \frac{M_{ii}}{M_T} \qquad \tilde{M}_{ii}^{(2)} = \frac{12(1 - \nu)M_{ii}}{\sigma_c\alpha Eh^2}$$
$$\tilde{Q}_\varphi^{(1)} = \frac{Q_\varphi R_c}{M_T} \qquad \tilde{Q}_\varphi^{(2)} = \frac{12(1 - \nu)R_cQ_\varphi}{\sigma_c\alpha Eh^2}$$
$$M_T = \frac{\alpha Eh^3S_c}{12(1 - \nu)R_c} \qquad \tilde{R}_2 = \frac{R_2}{R_c}$$
$$\tilde{s} = \frac{s}{R_c} \qquad \epsilon = \frac{h^2}{12R_c^2}$$
$$\lambda = \frac{R_2}{R_1}$$

in the augmented relations of Ref. 1, we obtain a set of equations for each type of temperature variation. They are: the equations of equilibrium

$$\tilde{Q}_\varphi^{(1,2)} = (d\tilde{M}_{\varphi\varphi}^{(1,2)}/d\tilde{s}) + (\tilde{M}_{\varphi\varphi}^{(1,2)} - \tilde{M}_{\theta\theta}^{(1,2)})\cdot\cot\varphi/\tilde{R}_2$$
$$\tilde{N}_{\phi\phi}^{(1,2)} = [\epsilon/(1 - \nu),\ 1]\cdot\tilde{Q}_\varphi^{(1,2)}\cdot\cot\varphi$$
$$\tilde{N}_{\theta\theta}^{(1,2)} = [\epsilon/(1 - \nu), 1]\cdot(d/d\tilde{s})\ (R_2\tilde{Q}_\varphi^{(1,2)})$$

the equation of compatibility

$$\tilde{V}^{(1,2)} = -\tilde{R}_2\cdot\frac{d\tilde{\epsilon}_\theta^{(1,2)}}{d\tilde{s}} + (\tilde{\epsilon}_\varphi^{(1,2)} - \tilde{\epsilon}_\theta^{(1,2)})\cdot\cot\varphi$$

and, the stress-strain, moment-curvature relations

$$[(1 - \nu^2)\tilde{N}_{\phi\phi}^{(1)}, (1 + \nu)\tilde{N}_{\varphi\varphi}^{(2)}] = \tilde{\epsilon}_\varphi^{(1,2)}\cdot(1 - \epsilon\lambda/\tilde{R}_2^2) + \nu\tilde{\epsilon}_\theta^{(1,2)}(1 - \epsilon/\tilde{R}_2^2) - (1 + \nu)\cdot(\tilde{S}, \tilde{\sigma}/\tilde{R}_2) + (\epsilon/\tilde{R}_2)\cdot[(d\tilde{V}^{(1,2)}/d\tilde{s}) + \nu\cdot\tilde{V}^{(1,2)}\cot\varphi/\tilde{R}^2]$$
$$[(1 - \nu^2)\cdot\tilde{N}_{\theta\theta}^{(1)}, (1 + \nu)\tilde{N}_{\theta\theta}^{(2)}] = \tilde{\epsilon}_\theta^{(1,2)}\cdot(1 - \lambda\epsilon/\tilde{R}_2^2) + \nu\tilde{\epsilon}_\varphi^{(1,2)}\cdot(1 - \lambda^2\epsilon/\tilde{R}_2^2) - (1 + \nu)(\tilde{S}, \lambda\tilde{\sigma}/\tilde{R}_2) + (\lambda\epsilon/\tilde{R}_2^2)\cdot[\tilde{V}^{(1,2)}\cot\varphi + \nu\tilde{R}_2\cdot(d\tilde{V}^{(1,2)}/d\tilde{s})]$$

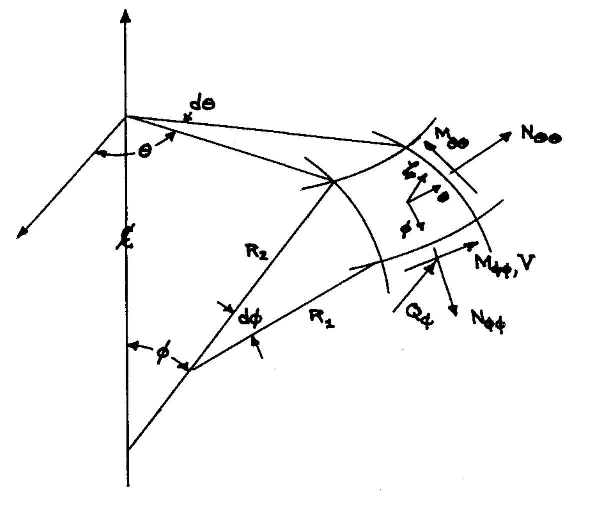

Fig. 1 Notation.

$$\tilde{M}_{\varphi\varphi}^{(1,2)} = -(\tilde{S}/\tilde{R}_2, \tilde{\sigma}) + (1, \epsilon)\cdot$$
$$[(d\tilde{V}^{(1,2)}/d\tilde{s}) + \nu\tilde{V}^{(1,2)}\cot\varphi/\tilde{R}_2 + (1-\lambda)\cdot\tilde{\epsilon}_\theta^{(1,2)}/\tilde{R}_2]/(1+\nu)$$
$$\tilde{M}_{\theta\theta}^{(1,2)} = -(\lambda\tilde{S}/\tilde{R}_2, \tilde{\sigma}) + (1, \epsilon)\cdot$$
$$[\nu\cdot(d\tilde{V}^{(1,2)}/d\tilde{s}) + \tilde{V}^{(1,2)}\cot\varphi/\tilde{R}_2 - (1-\lambda)\cdot\tilde{\epsilon}_\theta^{(1,2)}/\tilde{R}_2]/(1+\nu)$$

Implicit in the equations of equilibrium is the restriction that the shell is otherwise unloaded.

Approximate Solution for Middle Surface Heating

It follows from the preceding equations that all barred quantities are of order unity, save for the middle surface stresses that are of order (ϵ). Thus, it follows that a satisfactory solution is obtained by writing

$$\tilde{\epsilon}_\varphi^{(1)}, \tilde{\epsilon}_\theta^{(1)} = \tilde{S} \qquad \tilde{V}^{(1)} = -\tilde{R}_2\cdot\frac{d\tilde{S}}{d\tilde{s}}$$

$$(1+\nu)\cdot\tilde{M}_{\varphi\varphi}^{(1)} = (d\tilde{V}^{(1)}/d\tilde{s}) + \nu\tilde{V}^{(1)}\cot\varphi/\tilde{R}_2 - (\lambda+\nu)\tilde{S}/\tilde{R}_2$$
$$(1+\nu)\cdot\tilde{M}_{\theta\theta}^{(1)} = \tilde{V}\cot\varphi/\tilde{R}_2 + \nu\cdot(d\tilde{V}/d\tilde{s}) - (1+\lambda\nu)\tilde{S}/\tilde{R}^2$$

However, for $\tilde{S}$ = const = 1, say, we find $\tilde{V}^{(1)} = 0\ (\epsilon)$, so that the approximate solution becomes

$$\tilde{\epsilon}_\varphi^{(1)}, \tilde{\epsilon}_\theta^{(1)} = 1 \qquad (1+\nu)\cdot\tilde{M}_{\phi\phi}^{(1)} = -(\lambda+\nu)/\tilde{R}_2$$
$$(1+\nu)\tilde{Q}_\varphi^{(1)} = -\ [\tilde{R}_2\cdot(d\lambda/d\tilde{s}) - (1-\lambda^2)\cdot\cot\varphi]/\tilde{R}_2^{\,2}$$

The behavior of the solution in the neighborhood of the apex can be observed in the limiting case of a sphere of radius a by taking $\tilde{S} = 1, R_c = a$, i. e.,

$$\tilde{N}_{\phi\phi}^{(1)}, \tilde{N}_{\theta\theta}^{(1)}, \tilde{V}^{(1)}, \tilde{Q}_\varphi^{(1)} = 0$$
$$\tilde{M}_{\phi\phi}^{(1)}, \tilde{M}_{\theta\theta}^{(1)} = -1 \qquad \tilde{\epsilon}_\varphi^{(1)}, \tilde{\epsilon}_\theta^{(1)} = 1$$

Approximate Solution for a Temperature Gradient

For this case, all barred quantities are of order of magnitude unity, and hence a satisfactory solution is obtained by taking

$$\tilde{M}_{\varphi\varphi}^{(2)}, \tilde{M}_{\theta\theta}^{(2)} = -\tilde{\sigma} \qquad \tilde{Q}_\varphi^{(2)} = -\frac{d\tilde{\sigma}}{d\tilde{s}}$$

$$\tilde{N}_{\theta\theta}^{(2)} = -\frac{d}{d\tilde{s}}\left(\tilde{R}_2\cdot\frac{d\tilde{\sigma}}{d\tilde{s}}\right) \qquad \tilde{N}_{\phi\phi}^{(2)} = -\frac{d\tilde{\sigma}}{d\tilde{s}}\cdot\cot\varphi$$

$$\tilde{\epsilon}_\varphi^{(2)} = \tilde{\sigma}(1-\lambda\nu)/(1-\nu)\tilde{R}_2 + (\tilde{N}_{\phi\phi}^{(2)} - \nu\tilde{N}_{\theta\theta}^{(2)})/(1-\nu)$$
$$\tilde{\epsilon}_\theta^{(2)} = \tilde{\sigma}\cdot(\lambda-\nu)/(1-\nu)\tilde{R}_2 + (\tilde{N}_{\theta\theta}^{(2)} - \nu\tilde{N}_{\phi\phi}^{(2)})/(1-\nu)$$
$$\tilde{V}^{(2)} = -\tilde{R}_2\cdot(d\tilde{\epsilon}_\theta^{(2)}/d\tilde{s}) +$$
$$(1+\nu)\ [\tilde{\sigma}(1-\lambda)/\tilde{R}_2 + \tilde{N}_{\phi\phi}^{(2)} - \tilde{N}_{\theta\theta}^{(2)}]\cdot\cot\phi/(1-\nu)$$

However, for $\tilde{\sigma}$ = const = 1, say, we find

$$\tilde{Q}_\varphi^{(2)}, \tilde{N}_{\varphi\varphi}^{(2)}, \tilde{N}_{\theta\theta}^{(2)} = 0(\epsilon)$$

so that the approximate solution becomes

$$\tilde{\epsilon}_\varphi^{(2)} = (1-\lambda\nu)/\tilde{R}_2(1-\nu) \qquad \tilde{\epsilon}_\theta^{(2)} = (\lambda-\nu)/\tilde{R}_2(1-\nu)$$
$$(1-\nu)\tilde{V}^{(2)} = -\tilde{R}_2\ (d/d\tilde{s})\ [(\lambda-\nu)/\tilde{R}_2] +$$
$$(1+\nu)(1-\lambda)\cot\varphi/\tilde{R}_2$$

The behavior of the solution in the neighborhood of the apex can be observed in the limiting case of a sphere of radius a by taking $\tilde{\sigma} = 1, R_c = a$, i. e.,

$$\tilde{N}_{\phi\phi}^{(2)}, \tilde{N}_{\theta\theta}^{(2)}, \tilde{V}^{(2)}, \tilde{Q}_\varphi^{(2)} = 0$$
$$\tilde{M}_{\varphi\varphi}^{(2)}, \tilde{M}_{\theta\theta}^{(2)}, \tilde{\epsilon}_\varphi^{(2)}, \tilde{\epsilon}_\theta^{(2)} = 1$$

Concluding Remarks

As mentioned previously, the complete solution that satisfies a set of boundary conditions is obtained by superposing on the preceding particular solution a solution for arbitrary edge loading. If the shell is shallow, the results given in Ref. 2 are applicable in general, and Ref. 3 may be applied to spherical shells in particular. Alternatively, if the shell is not shallow, the method of asymptotic integration as proposed in Ref. 4 is applicable. The accuracy of the resulting composite solution depends, of course, on the region of the shell and on the accuracy of the edge-loading solution.

Finally, it can be shown that the preceding technique for constructing a particular solution is applicable to nonaxisymmetric temperature distributions provided that the circumferential variation may also be considered as slowly varying. For this more general problem, the boundary conditions may be satisfied by superposing a solution to the shallow shell equations, if applicable, or the solution proposed in Ref. 5.

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Zero Angle-of-Attack Sensor

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It is desirable to have under certain conditions a re-entry vehicle enter at zero angle of attack and maintain this angle during re-entry. In order to control flight at zero angle of attack, it is necessary to sense deviations from zero. One method for detecting angle of attack is shown in Fig. 1. Two σU transducers[1,2] are mounted on either face of a wedge. The signal from transducer 1 is less than the signal from transducer 2. A control system activates flaps or reaction jets to equalize the signals from the two transducers.

The σU transducers, which are described in Refs. 1 and 2, generate a voltage proportional to the product of electrical conductivity and local flow velocity. The relation between signal voltage e and σU is[1]

$$e = \int_{\text{ablation surface}}^{\text{shock wave}} \sigma U K dy \tag{1}$$

where K is the influence function for the transducer, and y is distance normal to the flow. Flight transducers with K (at y = 0) = 10 mv/(mho/m)(m/sec)(m) can be readily built.

To determine the change in signal for a change in angle of attack, σU for flow over a wedge was calculated. A free-stream velocity of V_∞ = 26,000 fps was chosen. Using Feldman's[3] tables, ρ_w, T_w, and V_w were obtained; the notation of the tables is used here. The velocity along the wedge V_w is equal to U.

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